Invariant Hamiltonian Quantization of General Relativity

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Abstract

The quantization of General Relativity invariant with respect to time-reparametrizations is considered. We construct the Faddeev-Popov generating functional for the unitary perturbation theory in terms of invariants of the kinemetric group of diffeomorphisms of a frame of reference as a set of Einstein's observers with the equivalent Hamiltonian description ($t' = t'(t)$, $x'^i = x'^i(t, x^1, x^2, x^3)$). The algebra of the kinemetric group has other dimensions than the constraint algebra in the conventional Dirac-Faddeev-Popov (DFP) approach to quantization.

To restore the reparametrization invariance broken in the DFP approach, the invariant dynamic evolution parameter is introduced as the zero Fourier harmonic of the space metric determinant. The unconstrained version of the reparametrization invariant GR is obtained. We research the infinite space-time limit of the Faddeev-Popov generating functional in the theory and discuss physical consequences of the considered quantization.

1. Introduction

Quantization of General Relativity (GR) on the level of the unitary [1] perturbation theory [2] was made by Faddeev and Popov [3, 4] and by DeWitt [5] in accordance with the general Hamiltonian theory for constrained systems formulated by Dirac [2, 6] and developed later by many authors (see e.g. monographs [7, 8, 9]). However, the construction of the unitary S-matrix in an infinite space-time is not enough to answer the questions: What is Quantum Gravity? and What is Quantum Universe with a finite measurable time of its existence and a finite volume of its space? Actual problems of the unification of elementary particle physics with General Relativity, cosmology of the Early Universe, in particular, the description of quantum processes at the beginning of the Universe require the generalization of these results [4, 5] for a finite space-time.

In the present paper we try to generalize the Faddeev-Popov-DeWitt construction of the unitary S-matrix for a finite space-time and to answer the above mentioned questions about the quantum version of GR.

A main problem of generalization of this type is the invariance of GR with respect to the group of general coordinate transformations. This group includes the kinemetric group of diffeomorphisms of a frame of reference as a set of Einstein's observers with the equivalent Hamiltonian description of GR [10]. Any Hamiltonian description of GR (classical or quantum) should be invariant with respect to transformations of this diffeomorphism group including reparametrizations of the coordinate time.

Therefore, the coordinate time should be excluded from the reparametrization invariant Hamiltonian dynamics. Recall, that just the coordinate time is considered as the time of evolution in both the Faddeeev-Popov-DeWitt unitary S-matrix and the Dirac Hamiltonian approach to GR [2, 6, 4].

One of constructive ideas for restoration of the reparametrization-invariance of the Hamiltonian dynamics in GR is the introduction of the internal evolution parameter as one of dynamic variables of the extendend phase space [11, 12, 13, 14, 15, 16, 17, 18]. In the present paper, this idea is used to construct the reparametrization-invariant generalization of the Faddeeev-Popov-DeWitt unitary S-matrix in GR. We fulfil the invariant Hamiltonian quantizing GR with a dynamic evolution parameter identified with the zero-Fourier harmonic of the space-metric determinant [16, 17].

This quantization is considered in the context of the Dirac perturbation theory [2] in its simplest version, without nontrivial topology, black holes, and surface terms in the Einstein-Hilbert action.

The contents of the paper are the following. In Section 2, we recall the conventional Dirac-Faddeev-Popov quantization and present the invariant Hamiltonian approach to GR corresponding to the diffeomorphism group of the Hamiltonian description. Section 3 is devoted to the formulation of the Dirac perturbation theory in GR with the invariant description of classical dynamics, in the reduced phase space, and of the measurable interval. In Section 4, we consider the global sector of the invariant dynamics and define two different standards of measurement of the invariant intervals: the absolute standard for an Einstein observer and the relative standard for a Weyl observer who can measure only a ratio of the lengths of two intervals and who treats GR as a scalar version of the Weyl conformal-invariant theory [17, 19]. Section 5 is devoted to the construction of quantum physical states of the Dirac perturbation theory [2]. In Section 6, we construct S-matrix for Quantum Universe and research the conditions of validity of the conventional quantum field theory in the infinite space-time limit.

2. Invariant Hamiltonian dynamics of GR

2.1. The Dirac-Faddeev-Popov Quantization

To state the problem considered in the present paper, we recall the Dirac-Faddeev-Popov (DFP) quantized General Relativity [2, 3, 4, 20] which is given by the Einstein-Hilbert action

$$W^{gr}(g|\mu) = \int d^4x[-\sqrt{-g}\frac{\mu^2}{6}R(g)] \quad (\mu^2 = M^2_{Planck}\frac{3}{8\pi}) \tag{1}$$

and by a measurable interval

$$(ds)^2_e = g_{\alpha\beta}dx^\alpha dx^\beta. \tag{2}$$

They are invariant with respect to general coordinate transformations

$$x_\mu \to x'_\mu = x'_\mu(x_0, x_1, x_2, x_3). \tag{3}$$

The Hamiltonian description of GR is fulfilled in the frame of reference with the Dirac-ADM $3+1$ parametrization of the metric components [21]

$$(ds)^2_e = g_{\mu\nu}dx^\mu dx^\nu = N^2dt^2 - {}^{(3)}g_{ij}\breve{dx}^i\breve{dx}^j \qquad (\breve{dx}^i = dx^i + N^i dt), \tag{4}$$

and it is given by the first-order representation of the initial action (1)

$$W^E = \int\limits_{t_1}^{t_2} dt \int d^3x[-\pi_{ij}\dot{q}^{ij} - N_q\mathcal{H} - N^i\mathcal{P}_i + \text{surf.terms.}] \tag{5}$$

obtained by Dirac [2] in terms of harmonical variables

$$q^{ik} = ||g||g^{ik}, \quad N_q = N||g||^{-1/3}, \qquad (||g|| = \det({}^{(3)}g_{ij}), \quad q = ||q^{ij}||), \tag{6}$$

$$\mathcal{H} = \frac{6}{\mu^2}q^{ij}q^{kl}[\pi_{ik}\pi_{jl} - \pi_{ij}\pi_{kl}] + \frac{\mu^2 q^{1/2}}{6}{}^{(3)}R, \tag{7}$$

$$\mathcal{P}_i = 2[\nabla_k(q^{kl}\pi_{il}) - \nabla_i(q^{kl}\pi_{kl})]. \tag{8}$$

We omit in equation (5) surface terms and use a finite space-time

$$\int d^3x = V_0, \qquad t_1 < t < t_2. \tag{9}$$

According to the Dirac general Hamiltonian theory [2, 6], the time components of the metric N_q, N^i are considered as the Lagrange multipliers, and six space components q^{ij}, as dynamic variables. Four local Einstein equations for the time components of metric

$$\phi_0 := \mathcal{H} = 0; \qquad \phi_i := \mathcal{P}_i = 0. \tag{10}$$

are treated as the first class constraints which remove four variables from the extended phase space. These four constraints should be supplemented by four local gauges which remove their canonical partners. Dirac [2] imposed the "gauge" constraints

$$\chi_0 := \pi := q^{ij}\pi_{ij} = 0, \qquad \chi^k := \partial_l(q^{-1/3}q^{lk}) = 0 \tag{11}$$

to express the initial action in terms of two independent dynamic variables of the metric.

The algebra of commutation relations of all constraints forms the Faddeev-Popov (FP) determinant [3, 4] which restores the unitarity of S-matrix in quantum theory.

This determinant for constraints (10), (11) has been computed in the monograph [20]

$$det\,\{\phi_\mu, \chi_\nu\} = detA detB, \tag{12}$$

where A and B^i_k are operators acting by the rules

$$Af = q^{ij}\nabla_i\nabla_j f + q^{1/2(3)}Rf; \tag{13}$$

$$B^i_k\eta_i = q^{-1/3}q^{lj}[\delta^i_k\partial_l\partial_j + \frac{1}{3}\delta^i_l\partial_j\partial_k]\eta_i. \tag{14}$$

According to the Faddeev-Popov prescription [3], the generating functional of Green functions has the form of the functional integral

$$Z_{DFP}[t_1|t_2] = \int D(q, \pi, N_q, N^k)[FP]_s[FP]_t \exp\left\{iW^E(g|\mu) + \text{sources}\right\}, \tag{15}$$

where

$$D(q, \pi, N_q, N^k) = \prod_x \left(\prod_{i<k} dq^{ik}d\pi_{ik}\prod_{j=1}^{3} dN^j dN_q\right) \tag{16}$$

and

$$[FP]_s = \delta(\chi_j))detB. \tag{17}$$

$$[FP]_t = \delta(\chi_0))detA. \tag{18}$$

are the space and time parts of the FP determinant. This functional and its gauge-equivalent versions are the foundation of the quantum field theory approach to General Relativity in the framework of the perturbation theory formulated in the infinite space-time limit.

2.2. Diffeomorphism group of the Hamiltonian description

To extract any physical information from relativistic systems, one should point out a frame of reference [10]. The latter means to answer the questions: Which quantities can an observer measure? How do these quantities connect with the metric components? How do results of measurements depend on a state of motion of an observer? The simplest example is the description of the energy spectrum of a relativistic particle in Special Relativity (SR) with the Poincare group of symmetry. There are two distinguished frames of the Hamiltonian description: the rest frame of an observer, and the comoving frame. In both the cases, to solve the physical problem and to obtain the spectrum, it is sufficient to restrict SR by only the subgroup of the Poincare group which exludes the pure Lorentz transformations. The latter are needed to answer the last question: How does the spectrum in the rest frame connect with the one in the comoving frame?

The main assertion of the present paper is the following: for the Hamiltonian description of GR in a definite frame of reference, it is sufficient to restrict GR by the group of diffeomorphisms of this frame.

Recall that the Hamiltonian formulation (5) is based on the possibility to introduce a set of the three-dimensional space-like hyperspaces (4) numerated by the time-like coordinate in the four-dimensional manifold of the world events. This set can be defined within transformations of a kinemetric subgroup of the group of general coordinate transformations (3) [10, 16, 17, 18]

$$t \to t' = t'(t); \qquad x_i \to x'_i = x'_i(t, x_1, x_2, x_3), \tag{19}$$

which includes one global function (the time reparametrizations $t'(t)$) and three local ones ($x'_i(t,x)$). This is the group of diffeomorphisms of a set of Einstein's observers with the equivalent Hamiltonian dynamics (5). This continuum of "observers" with the diffeomorphism group (19) is called the kinemetric frame of reference [10]. Action (5) can be written in terms of the diffeomorphism invariants

$$W^E(g|\mu) = \int\limits_{t_1}^{t_2} dt \int d^3x[-\pi_{ij}D_tq^{ij} - N_q\mathcal{H}], \tag{20}$$

including invariant differentials

$$dtD_tq^{ij} = dt[\dot{q}^{ij} + q^{ik}\nabla_kN^j + q^{jk}\nabla_kN^i - 2q^{ij}\nabla_kN^k]. \tag{21}$$

The dimensions of diffeomorphism group (19) do not coincide with the dimensions of the Dirac-Faddeev-Popov (DFP) algebra of constraints.

It is easy to see that, in the case of a finite time interval ($t^1 < t < t^2$), the DFP generating functional (15) breaks the invariance with respect to the global reparametrization of the coordinate time (19).

On the other hand, a local transformation of the coordinate time $t' = t'(t,x)$ goes beyond the scope of this group (19).

Thus, the generating functional (15) takes into account the symmetry which is absent in the diffeomorphism group (19) and it breaks the symmetry contained in this group.

Below we show that three gauges (11) $\chi^k = 0$ are sufficient to remove all local ambiguities from the Hamiltonian dynamics, so that the fourth local gauge can contradict the Hamiltonian equations of motion. Only the space integral from the equation for the lapse-function could be considered as the standard first class constraint (accompanied by the second class one, i.e. gauge) in agreement with the diffeomorphism group of the Hamiltonian description (19).

2.3. The invariant Hamiltonian scheme

To restore the time-reparametrization invariance, we introduce the internal evolution parameter as the zero Fourier harmonic $\varphi_0(t)$ of the space metric determinant logarithm [16]. This evolution parameter can be extracted by the conformal-type transformation of the metric

$$g_{\alpha\beta}(t,x) = (\frac{\varphi_0(t)}{\mu})^2\bar{g}_{\alpha\beta}(t,x). \tag{22}$$

The local part of momentum of the space metric determinant $\bar{\pi}$ and its motion equation equivalent $\bar{k}$

$$\bar{\pi}(t,x) := \bar{q}^{ij}\bar{\pi}_{ij}, \qquad \bar{k}(t,x) := \frac{\bar{q}_{ij}D_t\bar{q}^{ij}}{\bar{N}_q} = \frac{D_t\log\bar{q}}{\bar{N}_q} \tag{23}$$

are given in the class of functions with the non-zero Fourier harmonics, so that

$$\int d^3x\bar{\pi}(t,x) = 0, \qquad \int d^3x\bar{k}(t,x) = 0. \tag{24}$$

Using the transformational properties of the curvature $R(g)$ with respect to the transformation (22) it is easy to obtain the action (1) in the form

$$W^E(g|\mu) = W^E(\bar{g}|\varphi_0) - \int\limits_{t_1}^{t_2} dt\varphi_0\frac{d}{dt}(\dot{\varphi}_0V) \qquad (V = \int d^3x\bar{N}_q^{-1}), \tag{25}$$

with the same number of variables. The Hamiltonian form of this action is

$$W^E(q|\mu) = \int\limits_{t_1}^{t_2} dt \left(\int d^3x[-\bar{\pi}_{ij}\dot{\bar{q}}^{ij} - \bar{N}_q\bar{\mathcal{H}} - N^i\bar{\mathcal{P}}_i] - \dot{\varphi}_0 P_0 + \frac{P_0^2}{4V} \right), \tag{26}$$

where the densities of the local excitations $\bar{\mathcal{P}}_i$ and $\bar{\mathcal{H}}$ repeat the conventional Einstein ones (8) and (7) where the Planck constant μ is replaced by the internal evolution parameter φ_0 and q, π are replaced by $\bar{q}, \bar{\pi}$

$$\bar{\mathcal{P}} = \mathcal{P}(q, \pi \to \bar{q}, \bar{\pi}); \quad \bar{\mathcal{H}} = \mathcal{H}(q, \pi \to \bar{q}, \bar{\pi};\ \mu \to \varphi_0). \tag{27}$$

We shall consider action (26) as one of the kinemetric invariant versions of the Hamiltonian dynamics with the global variables which allow us to extract the time-reparamerization invariant physical consequences in accordance with the diffeomorphism group (19).

2.4. Unconstrained form of GR

The unconstrained form of GR is obtained by explicit resolving the constraints. The space constraints

$$\frac{\delta W^E}{\delta N^k} = 0 \Rightarrow \bar{\mathcal{P}}_k = 0 \tag{28}$$

and the diffeomorphism group (19) allow us to remove from the extended phase space three local components of the graviton field by fixing the gauge (11)

$$\chi^k = \partial_i(\bar{q}^{-1/3}\bar{q}^{ik}) = 0. \tag{29}$$

Constraints (28), and (29) can be explicitly solved by the decomposition of momenta $\bar{\pi}_{ij}$ into the transverse part $\bar{\pi}^T_{ij}$ and longitudinal components f_k

$$\bar{\pi}_{ij} = \bar{\pi}^T_{ij} + \bar{q}^{-1/3}[\partial_i f_j + \partial_j f_i - \frac{2}{3}\bar{q}_{ij}\bar{q}^{lk}\partial_l f_k] \qquad (\ \partial^i[\bar{q}^{1/3}\bar{\pi}^T_{ij}] = 0\ ). \tag{30}$$

Substitution of this decomposition into constraint (28) leads to the equation for the longitudinal component f_k

$$\bar{\mathcal{P}}_i(\bar{\pi}, \bar{q}) = \bar{\mathcal{P}}_i(\bar{\pi}, \bar{q}) - \frac{1}{2}B^k_i f_k = 0 \Rightarrow B^k_i f_k = \frac{1}{2}\bar{\mathcal{P}}_i(\bar{\pi}^T, \bar{q}) \tag{31}$$

in the class of local functions with the non-zero Fourier harmonics where the reverse operator $(B^{-1})^i_k$ (14) exists.

Let us consider the equation for the lapse function

$$\bar{N}_q \frac{\delta W^E}{\delta \bar{N}_q} = 0 \Longrightarrow \frac{P_0^2}{4V^2\bar{N}_q} - \bar{N}_q\bar{\mathcal{H}} = 0. \tag{32}$$

The integration of of equation (32) over the space coordinates determines the global momentum P_0 in action (26)

$$\frac{1}{V}\left(\int d^3x \bar{N}_q \frac{\delta W^E}{\delta \bar{N}_q} \right) = 0 \Longrightarrow (P_0)_\pm = \pm 2\sqrt{VH} \equiv \pm H^R \qquad (\ H = \int d^3x' \bar{N}_q \bar{\mathcal{H}}\ ). \tag{33}$$

as the functional of all other variables. It is the generator of evolution with respect to the dynamic evolution parameter φ_0. Thus, the global part of equation (32) removes the global momentum P_0, in the correspondence with the diffeomorphism group (19). The orthogonal to (33) (local) part of the same equation (32)

$$\left(\bar{N}_q \frac{\delta W^E}{\delta \bar{N}_q} - \frac{1}{V}\int d^3x \bar{N}_q \frac{\delta W^E}{\delta \bar{N}_q}\right) = 0 \implies \bar{N}_q \bar{\mathcal{H}} - \frac{H}{\bar{N}_q V} = 0, \tag{34}$$

together with six equations for the transverse space metric components $\bar{\pi}^T_{ij}$, $\bar{q}^{ij}$, determines the lapse function $\bar{N}_q$ with an arbitrary time-dependent factor $\beta(t)$

$$\bar{N}_q(t,x) = [\sqrt{\bar{\mathcal{H}}(t,x)}]^{-1}\beta(t). \tag{35}$$

This factor represents the Lagrange multiplier. The generator of the dynamic evolution (33)

$$H^R = 2\sqrt{VH} = 2\int d^3x \sqrt{\bar{\mathcal{H}}(t,x)} \tag{36}$$

does not depend on this factor $\beta(t)$.

In accordance with the diffeomorphism group (19), we consider as constraint only equation (33) which is the equation for the Lagrange multiplier $\beta(t)$.

The global constraint (33) has two solutions which correspond to two reduced systems with the actions

$$W^R_{\pm} = \int\limits_{\varphi_1=\varphi_0(t_1)}^{\varphi_2=\varphi_0(t_2)} d\varphi \left\{\left(-\int d^3x \bar{\pi}^T_{ij}\partial_\varphi \bar{q}^{ij}\right) \mp H^R\right\} \tag{37}$$

where H^R is the Hamiltonian of evolution (36) of the reduced phase space variables $(\bar{\pi}^T_{ij},\ \bar{q}^{ij})$ with respect to the dynamic evolution parameter $\varphi = \varphi_0$.

Following to Dirac [6], we call the sector of the reduced phase space described by action (37) the Dirac "observables". These variables are kinemetric invariants by the construction. The equations of motion of the reduced unconstrained system are

$$\frac{\delta W^R}{\delta \bar{\pi}_{ijT}} = 0 \Rightarrow \frac{\partial \bar{\pi}_{ijT}}{\partial \varphi} = \pm \frac{\delta H^R}{\delta \bar{q}^{ij}}, \tag{38}$$

$$\frac{\delta W^R}{\delta \bar{q}^{ij}} = 0 \Rightarrow \frac{\partial \bar{q}^{ij}}{\partial \varphi} = \mp \frac{\delta H^R}{\delta \bar{\pi}^T_{ij}}. \tag{39}$$

Solutions of equations (38), (39) determine the dependence of the Dirac observables with on the dynamic evolution parameter φ.

The main problem is to construct the time-reparametrization invariant Faddeev- Popov generating functional for the unitary perturbation theory.

3. The invariant version of the Dirac perturbation theory

The reparametrization-invariant version of the perturbation theory begins from the nonperturbative background metric with the homogeneous part of the space metric (22) (which gives the dynamic evolution parameter) and the global component of the lapse function $\bar{N}_0$ (35) which defines the reparametrization-invariant conformal time

$$\bar{N}_q = N_0 \bar{N}; \qquad dT = N_0(t)dt \qquad (dT' = N_0' dt' = dT). \tag{40}$$

For the local part of metric, we use the version of the Dirac perturbation theory [2]

$$\bar{q}^{1/3}\bar{q}_{ij} = \delta_{ij} + h^T_{ij}; \quad \bar{q}^{1/3} = 1 + 4z + ...; \qquad \bar{N} = 1 + \nu + ... \tag{41}$$

Asymptotic states will be considered in the neglect of interactions, in accordance with the standard suppositions of quantum field theory.

In the lowest order of this theory equation (34) determines $z = (\log \bar{q})/12$

$$-\frac{2\varphi^2}{3}\Delta z = \bar{\mathcal{H}}_0 - \rho_0 \qquad (\rho_0 = \frac{\int d^3x \bar{\mathcal{H}}_0}{V_0}; \qquad V_0 = \int d^3x \ ) \ . \tag{42}$$

A solution of this equation recalls the FP gauge [4] where the internal evolution parameter is changed by μ and instead of the massive matter we have the non-zero Fourier harmonic part of the Hamiltonian for two transverse and trace-less gravitons

$$\bar{\mathcal{H}}_0 := \frac{6}{\varphi^2}(\pi^T)^2_{ij} + \frac{\varphi^2}{24}(\partial_k(h^T)_{ij})^2 := \varphi^{-2}\bar{\mathcal{H}}_K + \varphi^2\bar{\mathcal{H}}_R. \tag{43}$$

The solutions of equation (42) is usually treated as the Newton interaction of particles, i.e. of the transverse and trace-less gravitons h^T_{ij} which form the asymptotical physical states, similar to photons in QED. These transverse gravitons h^T_{ij} are considered in paper [18] in context of the invariant Hamiltonian quantization. The trace component of the graviton momentum $\bar{q}^{ij}\bar{\pi}_{ij} = p_z/12$ disappears from the kinetic part of the Hamiltonian (43) $\bar{\mathcal{H}}_K$ as a result of the solution of the space constraints. Nevertheless, the momentum p_z is not equal to zero as it follows from the equations of the initial extended system for p_z

$$2\varphi^2(z' - \partial_k\bar{N}_k) = p_z \qquad (z' = \frac{\partial z}{\partial T} = \sqrt{\rho_0}\partial_\varphi z), \tag{44}$$

and for $\bar{N}^k = N^k/N_0$

$$-\frac{2\varphi^2}{3}(\partial_k z' - \Delta\bar{N}_k) = \mathcal{P}^T_k := (\pi^T)_{ij}\partial_k(h^T)_{ij}, \tag{45}$$

in contrast with the Dirac gauge [2]. The perturbation part of the lapse function ν is determined from the motion equation of the reduced system for z unambiguously

$$\nu = \frac{2\varphi^2}{3}\Delta z \rho_0^{-1} \ . \tag{46}$$

One can see that the range of applicability of the Dirac perturbation theory (41) is the region where derivatives are far less than the internal evolution parameter $\Delta f/f << \varphi^2$.

In the opposite limit $\varphi_0 \rightarrow 0$, we got the local version of the model of an anisotropic universe considered by Misner [22].

4. Measurable quantities

4.1. Geometry

The dynamic sector of the unconstrained GR restricted by the Dirac "observables" (37), (38), (39) is not sufficient to determine evolution of the Einstein invariant interval

$$(ds_e)^2 = (\frac{\varphi}{\mu})^2\bar{q}^{1/6}(ds_c)^2, \tag{47}$$

here ds_c is the conformal invariant interval

$$(ds_c)^2 = \bar{q}^{1/3}[dT^2\bar{N}^2 - \bar{q}_{ij}\breve{d}x^i\breve{d}x^j]; \quad (\breve{d}x^j = dx^j + \bar{N}^j dT) \tag{48}$$

which does not depend on the global variable φ. These intervals characterize the measurable geometry of the space-time and contain the shift vector $\bar{N}^k$ and invariant time parameter (40). The latter is well-known in the classical Friedmann cosmology [14, 17] as the conformal time connected with the world Friedmann time by the relation

$$dT_f = \frac{\varphi(T)}{\mu} dT. \tag{49}$$

Measurable geometrical quantities go out from the set of the Dirac "observables", and can be determined by invariant equations of the initial extended system for the global variables P_0, φ and the local ones $\bar{\pi}_{jl}$ which are omitted by the reduced action (37). In particular, the evolution of the Universe is not also included in the dynamic sector of the Dirac "observables".

4.2. Evolution of a universe

The evolution of a universe is the dependence of the measurable time (40) on the internal evolution parameter given by the equation of the extended system

$$\frac{\delta W^E}{\delta P_0} = 0 \Rightarrow \left(\frac{d\varphi}{dT}\right)_\pm = \frac{(P_0)_\pm}{2V} = \pm\sqrt{\rho(\varphi)}; \quad \rho = \frac{\int d^3x \bar{\mathcal{H}}}{\bar{V}_0} = \frac{\bar{H}}{\bar{V}_0}; \tag{50}$$

The integral form of the last equation

$$T(\varphi_0) = \int\limits_0^{\varphi_0} d\varphi \rho^{-1/2}(\varphi). \tag{51}$$

is well-known as the Friedmann-Hubble law in the Friedmann-Robertson-Walker cosmology. It is natural to call the Hamiltonian $\bar{H}$ the "measurable" one, as it determines the evolution of the Dirac observables with respect to the measurable time T

$$f' := \frac{\partial f}{\partial T} = \sqrt{\rho}\partial_\varphi f = \{\bar{H}, f\}. \tag{52}$$

Another global equation of the extended system

$$\frac{\delta W^E}{\delta \varphi} = 0 \Rightarrow P_0' = V \frac{d}{d\varphi}\rho(\varphi) \tag{53}$$

leads to the conservation law for the measurable Hamiltonian $\bar{H}$ [16]

$$\varphi^{-2}\bar{H}_K' + \varphi^2 \bar{H}_R' = 0, \tag{54}$$

where symbols K, R mean the kinetic and potential parts (see eq. (43)). The shift vector is determined by the equation

$$\frac{\delta W^E}{\delta \bar{\pi}_{jl}} = 0 \Rightarrow \partial_T \bar{q}^{jl} + \nabla^j \bar{N}^l + \nabla^l \bar{N}^j = \frac{12\bar{N}}{\varphi^2}(\bar{q}^{ij}\bar{q}^{kl}\bar{\pi}_{ik} - \bar{q}^{jl}\bar{\pi}). \tag{55}$$

4.3. Standards of measurement

As it was shown in papers [16, 17], GR with the Einstein-Hibert action can be also treated as the scalar version of the Weyl conformal theory [19] with the scalar field Φ_w considered as the measure of a change of the length of a vector in its parallel transport. In this case the role of the metric scale field $\varphi_g = \mu q^{1/12}$ in GR is played by the Lichnerowich [23, 11] conformal invariant variable $\varphi_c = \Phi_w q^{1/12}$ of the scalar field. Dynamics of both the Einstein GR and the Weyl theory is the same (including the matter sector where the scalar field forms masses of fermion and boson fields), but not standards of measurement. An Einstein observer measures the absolute lengths $(ds)_e$, while a Weyl observer can measure only the ratio of lengths of two vectors $(ds)_w = (ds_1)_e/(ds_2)_e = (ds_1)_c/(ds_2)_c$ which is conformal-invariant. Thus, a classical state of the universe in GR (with the Einstein-Hilbert action) is determined both by the dynamic sector of the Dirac "observables" in the reduced phase space and the geometrical sector of "measurables"; the latter are determined not only by invariant dynamics of the Einstein-Hilbert action , but also by standards of measurements.

The same GR dynamics corresponds to different cosmological pictures for different observers: an Einstein observer, who supposes that he measures an absolute interval, obtains the Friedmann-Robertson-Walker (FRW) cosmology where the red shift is treated as expansion of the universe; a Weyl observer, who supposes that he measures a relative interval D_c, obtains the Hoyle-Narlikar cosmology [24]. The red shift and the Hubble law in the Hoyle-Narlikar cosmology [24]

$$Z(D_c) = \frac{\varphi(T)}{\varphi(T - D_c/c)} - 1 \simeq \mathcal{H}^c_{Hub} D_c/c; \qquad \mathcal{H}^c_{Hub} = \frac{1}{\varphi(T)}\frac{d\varphi(T)}{dT} = \frac{\sqrt{\rho(T)}}{\varphi(T)} \tag{56}$$

reflect the change of the size of atoms in the process of evolution of masses [24, 15, 16, 17].

Equation (56) gives the relation between the present-day value of the scalar field and cosmological observations (the density of matter and the Hubble parameter)

$$\varphi(T) = \frac{\sqrt{\rho(T)}}{\mathcal{H}^c_{Hub}(T)}. \tag{57}$$

Note that the present-day observational data [27] on the matter density

$$\rho = \rho_b = \Omega_0 \rho_{cr}; \qquad 0.1 < \Omega_0 < 2 \qquad (\rho_{cr} = \frac{3\mathcal{H}^c_{Hub}}{8\pi} M^2_{Pl}) \tag{58}$$

give the value of the dynamic evolution parameter which coincides with the Newton constant (or the Planck mass)

$$\varphi_0(T_0) = \mu \Omega_0^{1/2}. \tag{59}$$

Both the standards of measurement of the present day value of φ_0 in observational cosmology give the value of the Planck mass (59). Nevertheless, only for the relative standard of a Weyl observer, local measurements of the invariant interval do not depend on the parameters of global evolution of the universe.

5. "Measurable" Quantum Universe

We calculate the generating functional for the unitary perturbation theory as the S-matrix element in the standard interaction representation applied in quantum field theory

$$S[\varphi_1, \varphi_2] = \alpha^+ < out\ (\varphi_2)|T \exp\left\{-i\int\limits_{\varphi_1}^{\varphi_2} d\varphi (H^R_I)\right\}|(\varphi_1)\ in > +$$

$$\alpha^- < out\ (\varphi_1)|\widetilde{T} \exp\left\{+i\int\limits_{\varphi_1}^{\varphi_2} d\varphi(H_I^R)\right\}|(\varphi_2)\ in >,$$

where T, $\widetilde{T}$ are symbols of ordering and anti-ordering, H_I^R is the Hamiltonian of interaction of the reduced system

$$H_I^R = H^R - H_0^R \tag{60}$$

H^R is the reduced Hamiltonian defined by equation (36), H_0^R is a free part of this Hamiltonian H^R in the perturbation theory (41), (43), and $|(\varphi)\ in(out) >$ is the in (out) - state of Quantum Universe which satisfies the Schrödinger equation

$$\frac{d}{id\varphi}|(\varphi)\ in(out) >= H_0^R|(\varphi)\ in(out) > . \tag{61}$$

As we have seen above, the description of both the Dirac dynamics (in the reduced phase space) and the measurable geometry, i.e. the invariant interval

$$(ds_c)^2 = dT^2 - (\delta_{ij} + h_{ij})dx^i dx^j \tag{62}$$

can be given only by the constrained system with the extended action

$$W_0^E = \int\limits_{t_1}^{t_2} dt \left\{\left[\int d^3x \pi_{ij}^T \dot{h}_{ij}^T\right] - P_0\dot{\varphi} - N_0\left[-\frac{P_0^2}{V_0} + \bar{H}_0\right]\right\}, \tag{63}$$

where

$$\bar{H}_0 = \int d^3x \left(\frac{6(\pi_{(h)}^T)^2}{\varphi^2} + \frac{\varphi^2}{24}(\partial_i h^T)^2\right); \quad (h_{ii}^T = 0; \quad \partial_j h_{ji}^T = 0), \tag{64}$$

is the "measurable" Hamiltonian of "free" gravitons. This action includes the world time interval $dT = N_0 dt$ measured by an Weyl observer.

The dependence of the world time on the internal evolution parameter φ is treated as evolution of a classical Universe. Quantization of the extended constrained system with the "free" gravitons (63) was performed in paper [18] where the holomorphic variables of "particles" (a^+, a) were defined as variables which diagonalize the measurable Hamiltonian (64), and "quasiparticles" (b^+, b), as variables which diagonalize the classical and quantum equations of motion and lead to the equivalent oscillator-like system with the set of conserved "quantum numbers". For the latter system there is the canonical transformation [25, 26, 18] of the extended system (63) to a new set of variables

$$(a^+, a|P_0, \varphi) \Rightarrow (b^+, b|\Pi, \eta), \tag{65}$$

so that the new internal evolution parameter η coincides, in the equation of motion for the new momentum Π, with the invariant time measured by a Weyl observer in the comoving frame of references

$$\frac{\delta W_0^E}{\delta \Pi} = 0 \Rightarrow d\eta = N_0 dt = dT. \tag{66}$$

Thus, after the canonical transformations the states of the measurable Quantum Universe (with a conserved number of "quasiparticles") are determined by the Schrödinger equation

$$\frac{d}{idT}|\varphi(T)\ in >= \sum_n \omega_b(n,T)\frac{1}{2}(b_n^+ b_n + b_n b_n^+)|\varphi(T)\ in >= E_0^R|\varphi(T)\ in >, \tag{67}$$

where n denotes a set of parameters of gravitons (projections of spins, and momenta), and E_0^R is an eigenvalue of the oscillator-like Hamiltonian.

The state of "nothing" is the squeezed vacuum (without quasiparticles) $b_n(T)|\varphi(T)\ in >= 0$. It was shown [18] that for small φ, and a large Hubble parameter, at the beginning of the Universe, the state of vacuum of quasiparticles leads to the measurable density

$$_b < \rho(a^+,\ a) >_b = \rho_0 \frac{1}{2}\left(\frac{\varphi_0^2}{\varphi^2(T)} + \frac{\varphi^2(T)}{\varphi_0^2}\right), \tag{68}$$

where φ_0 is the initial value, and $\rho_0 = \frac{1}{2}\sum\limits_n \omega_a(n)$ is the density of the Kasimir vacuum of "particles". The first term corresponds to the rigid state equation (in accordance with the classification of the standard cosmology) and it leads to the Kasner anisotropic stage $T_\pm(\varphi) \sim \pm\varphi^2$ (described by the Misner wave function [22]). ¿From the point of view of fields of matter for which φ forms masses, the negative solution $\varphi^2(T_-) < 0$ (anti-Universe) is not stable, in this stage. The second term of the squeezed vacuum density (68) leads to the stage with the inflation of the scale φ with respect to the time measured by a Weyl observer

$$\varphi(T) \simeq \exp(T\sqrt{2\rho_0}/\varphi_0).$$

It is the stage of intensive creation of "measurable particles". After the inflation, the Hubble parameter goes to zero, and gravitons convert into photon-like oscillator excitations with the conserved number of particles.

At the present-day stage, we can describe in- and out -states in terms of the "measurable" time T and the Hamiltonian $\bar{H}_0$ (64) where φ is changed by μ, in agreement with the data of the observational cosmology $\varphi(T_0) = \mu$ discussed above.

The internal evolution parameter can be connected with the time measured by an observer of a quantum state of the Universe $|out>$ in terms of the conserved quantum numbers of this state: energy E_{out} and density $\rho_{out} = E_{out}/V_0$

$$\frac{d\varphi}{dT} = \sqrt{\rho_{out}}. \tag{69}$$

It is natural to suppose that E_{out} is a tremendous energy in comparison with possible deviations of the free Hamiltonian in the laboratory processes

$$\bar{H}_0 = E_{out} + \delta H_0, \qquad < \mathrm{o}ut|\delta H_0|\mathrm{i}n ><< E_{out}. \tag{70}$$

6. Infinite volume limit of Quantum Gravity

We consider the infinite volume limit of the S-matrix element in terms of the measurable time T for the present-day stage $T = T_0$ taking into account only the contribution of the Universe $\alpha^+ = 1, \alpha^- = 0$

$$S[T_1 = T_0 - \Delta T | T_2 = T_0 + \Delta T] =< \mathrm{o}ut\ (T_2)|T \exp\left\{-i \int\limits_{\varphi(T_1)}^{\varphi(T_2)} d\varphi (H_I^R)\right\}|(T_1)\ \mathrm{i}n > . \tag{71}$$

One can express this matrix element in terms of the time measured by an observer of an out-state with the tremendous number of particles in the Universe using equation (69) and the approximation (70).

In the infinite volume limit, we get

$$d\varphi[H_I^R] = dT[\hat{F}\bar{H}_I + O(1/V_0) + O(1/E_{out})] \tag{72}$$

where $\bar{H}_I$ is the Hamiltonian of interaction in GR, and

$$\hat{F} = \sqrt{\frac{E_{out}}{E_{out} + \delta \bar{H}_0}} \tag{73}$$

is the multiplier which plays the role of a form factor for physical processes observed at the "laboratory" conditions when the cosmic energy E_{out} is much greater than the deviation of the free energy

$$\delta \bar{H}_0 = \bar{H}_0 - E_{out}; \tag{74}$$

due to creation and annihilation of real and virtual particles in the laboratory experiments. The measurable time of the laboratory experiments $T_2 - T_1$ is much smaller than the age of the Universe T_0, but it is much greater than the reverse "laboratory" energy δ, so that the limit

$$\int\limits_{T_1}^{T_2} \Rightarrow \int\limits_{-\infty}^{+\infty}$$

is valid. We can get the conventional quantum field theory representation of matrix element (71)

$$S[-\infty| + \infty] =< out|T \exp \left\{ -i \int\limits_{-\infty}^{+\infty} dT \bar{H}_I \right\} |in >, \tag{75}$$

if we neglect the form factor (73) which removes a set of ultraviolet divergences. This matrix element corresponds to the FP functional integral

$$Z_{QFT} \;=\; \int D(\bar{q}, \bar{\pi}, \bar{N}^k)[FP]_s \exp \left\{ i \bar{W}^E[\bar{q}|\mu] + \text{sources} \right\}. \tag{76}$$

where $\bar{W}^E[\bar{q}|\mu]$ is the initial action (1) in terms of the conformal-invariant time T for $\bar{N} = 1$ (40).

The main difference of the obtained generating functional from the Faddeev-Popov-DeWitt one [4, 5] is the absence of the fourth gauge which fixes the determinant of the space metric [4] or its momentum [2]. In both the cases, these gauges contradict the motion equations for these variables, as we have seen above, in the context of the Dirac perturbation theory [2].

The result (76) could be predicted from the very beginning, the problem was to show the range of validity of the conventional quantum field perturbation theory [3, 4] and its possibilities for solution of problems of the Early Universe.

The relativistic covariance of the considered scheme of quantization can be proved in the infinite space-time on the level of algebra of commutation relations of the generators of the Poincare symmetry in perturbation theory by analogy with QED [28].

¿From the point of view of the quantum field theory limit, the conformal variables and measurable quantities, including the conformal time, are favorable, and the Einstein General Relativity looks like a scalar version of the Weyl conformal invariant theory, where the Weyl scalar field forms both the Planck mass (in agreement with the present-day astrophysical data) and masses of elementary particles [17] (in agreement with the principle of equivalence).

In the Weyl theory, the Higgs mechanism of the formation of particle masses becomes superfluous and, moreover, it contradicts the equivalence principle, as, in this case, the Planck mass and masses of particles are formed by different scalar fields.

In the conformal theory [16, 17], we got the σ-version of the Standard Model [19] without Higgs particles, and with the prescription (73) to be free from the ultra-violet divergences for the precision calculations.

7. Conclusion

We have obtained the generalization of the unitary S-matrix in General Relativity [5, 4, 20] for a finite space-time in agreement with the group of invariance of the Hamiltonian dynamics in GR. This group contains reparametrizations of the coordinate time (t) and gauge transformations with three local parameters. We have shown that the solution of one global constraint (with respect to the zero Fourier harmonic of the space metric determinant φ_0) and three local constraints remove all ambiguities from the perturbation theory for transverse gravitons, so that the fourth gauge [2, 4] for fixation of the space metric determinant is superfluous and can contradict equations of motion.

As a result of the solution of these constraints, we got the unconstrained version of GR which describes the dynamics of the Dirac "observables" in the reduced phase space with the dynamic evolution parameter. Besides the unconsrained dynamics, the extended Hamiltonian GR contains the geometry of "measurable quantities" (which depend on all components of metric including those which cannot be defined by complete set of equations in the sector of the Dirac "observables").

The geometric sector of "measurable intervals" is a specific feature of GR which strongly distinguishes it from classical unconstrained systems where the dynamic evolution parameter coincides with the measurable time.

In particular, the evolution of the universe is the evolution of the Dirac sector of "observables" (together with their dynamic evolution parameter) with respect to the "measurable" interval (including the invariant proper time), and this "measurable" evolution goes beyond the scope of the sector of the Dirac "observables". This fact is the main difficulty for the standard quantization.

To emphasize the autonomy of the "measurable" geometrical sector in GR, we pointed out two different standards of measurement (relative and absolute) which correspond to two theories with the same dynamics: GR and the scalar version of the Weyl geometry of similarity (with a scalar field as the measure of a change of the length of a vector in its parallel transport). In terms of the conformal invariant variables, actions of both these theories coincide, but the measurable intervals are different. An Einstein observer (who measures lengths by the absolute standard) sees the Friedmann-Robertson-Walker evolution of a universe, while a Weyl observer (who treats the determinant of the three-dimensional metric multiplied by the Planck constant as a measure of a change of the length of a vector in its parallel transport) sees the Hoyle-Narlikar evolution.

We have considered the phenomenon of Quantum Universe mainly "with a view to its measurement describing the methods of measurement and defining the standards on which they depend" [29].

The phenomenon of Quantum Universe can be described by two measurable quantities: the time in the comoving frame, and the red shift of spectral lines of the cosmic object atoms in terms of the dynamic evolution parameter (i.e. the scale factor). Both these quantities determine the background metric of the considered perturbation theory for the unconstrained GR and measurable density.

Now, we can define the Quantum Universe as the universe filled by "free" quantum fields in the space-time with the considered background metric and standard of the measurement of the invariant time intervals. The evolution of the Quantum Universe is expressed in terms of the measurable time by canonical transformations which convert the dynamic evolution parameter into the measurable time and the variables of particles (diagonalizing the measurable density) into the quasiparticles (diagonalizing equations of motion) with the squeezed vacuum.

The Quantum Gravity is the theory of S-matrix between the states of the Quantum Universe.

The infinite space-time limit of this S-matrix leads to the standard quantum field theory S-matrix provided the measurable time is the conformal time of a Weyl observer and General Relativity is the scalar version of the Weyl conformal invariant theory with the set of prediction, including

the Hoyle-Narlikar version of observational cosmology, where the physical reason of red-shift is changing masses of elementary particles in the process of evolution of the Universe, the cosmic mechanism of the formation of both the masses of elementary particles and the Planck mass by the Weyl scalar field (which does not contradict the present-day astrophysical data),

the squeezed vacuum inflation from "nothing" at the beginning of the Universe,

and the negative result of CERN experiment on the search of Higgs particles.

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